# Ulrich ideals with smallest number of generators

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based on the recent works jointly with

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# 1. Introduction

This talk is based on the recent researches below.

- N. Endo and S. Goto, *Ulrich ideals in numerical semigroup rings of small multiplicity*, arXiv:2111.00498
- N. Endo, S. Goto, S.-i. Iai, and N. Matsuoka, *Ulrich ideals in the ring* $k[[t^5, t^{11}]]$, arXiv:2111.01085

## Problem 1.1

**Determine all the Ulrich ideals in a given CM local ring.**

# What is an Ulrich ideal?

- In 1971, J. Lipman investigated **stable maximal ideal** in a CM local ring.
- In 2014, S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, K.-i. Yoshida modified the notion of stable maximal ideal, which they call an **Ulrich ideal**.

Let

- $(A, \mathfrak{m})$ be a CM local ring with $d = \dim A$.
- $\sqrt{I} = \mathfrak{m}$, $I$ contains a parameter ideal $Q$ of $A$ as a reduction
  (i.e. $I^{n+1} = QI^n$ for some $n \geq 0$)

**Definition 1.2 (Goto-Ozeki-Takahashi-Watanabe-Yoshida, 2014)**

We say that $I$ is an *Ulrich ideal of A*, if

(1) $I \supsetneq Q$, $I^2 = QI$, and

(2) $I/I^2$ is $A/I$-free.

Note that

- (1) $\iff$ $\mathrm{gr}_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is a CM ring with $\mathrm{a}(\mathrm{gr}_I(A)) = 1 - d$.
- If $I = \mathfrak{m}$, then (1) $\iff$ $A$ has minimal multiplicity $\mathrm{e}(A) > 1$.
- (2) and $I \supsetneq Q$ $\implies$ $\mathrm{pd}_A I = \infty$ (Ferrand, Vasconcelos, 1967)

Assume that $I^2 = QI$. Then the exact sequence

$$0 \to Q/QI \to I/I^2 \to I/Q \to 0$$

of $A/I$-modules shows

$$I/I^2 \text{ is } A/I\text{-free} \iff I/Q \text{ is } A/I\text{-free.}$$

Therefore, if $I$ is an Ulrich ideal of $A$, then

- $I/Q \cong (A/I)^{\oplus(\mu_A(I)-d)}$,
- $Q :_A I = I$ (i.e., $I$ is a good ideal of $A$),
- $(\mu_A(I) - d) \cdot \mathrm{r}(A/I) = \mathrm{r}(A)$

so that

$$d + 1 \leq \mu_A(I) \leq d + \mathrm{r}(A).$$

Hence, when $A$ is a Gorenstein ring,

every Ulrich ideal $I$ is generated by $d + 1$ elements (if it exists).

For every Ulrich ideal $I$ of $A$, we have

**Theorem 1.3 (Goto-Takahashi-T, 2015)**

$$\mathrm{Ext}_A^i(A/I, A) \text{ is } A/I\text{-free for } \forall i \in \mathbb{Z}.$$

Hence

$$\mu_A(I) = d + 1 \iff \text{G-dim}_A\, A/I < \infty.$$

This shows if $A$ is $G$-regular, then $\mu_A(I) \geq d + 2$.

Consequently, if $I$ is an Ulrich ideal of $A$ with $\mu_A(I) = d + 1$, then

- $A/I$ is Gorenstein $\iff$ $A$ is Gorenstein,
- $I$ is a totally reflexive $A$-module,
- $\mathrm{pd}_A\, I = \infty$, and

the minimal free resolution of $I$ has a very restricted form.

In what follows, assume $d = 1$ and $I$ is an Ulrich ideal of $A$ with $\mu_A(I) = 2$.

Write $I = (a, b)$, where $a, b \in A$ and $Q = (a)$ is a reduction of $I$.

By taking $c \in I$ with $b^2 = ac$, the minimal free resolution of $I$ has the form

$$\cdots \longrightarrow A^{\oplus 2} \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} A^{\oplus 2} \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} A^{\oplus 2} \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} I \longrightarrow 0$$

We then have $I = J$, once

$$\mathrm{Syz}_A^i(I) \cong \mathrm{Syz}_A^i(J) \quad \text{for some } i \geq 0$$

provided $I, J$ are Ulrich ideals of $A$. (GOTWY, 2014)

**Corollary 1.4 (GOTWY, 2014)**

*Suppose that $A$ is a Gorenstein ring. If $I, J$ are Ulrich ideals of $A$ with $\mathfrak{m}J \subseteq I \subsetneq J$, then $A$ is a hypersurface.*

Let $\mathcal{X}_A$ be the set of Ulrich ideals in $A$.

On the other hand

- If $A$ has finite CM representation type, then $\mathcal{X}_A$ is finite. (GOTWY, 2014)
- Suppose that $\exists$ a fractional canonical ideal $K$. Set $\mathfrak{c} = A : A[K]$.

  If $A$ is a non-Gorenstein almost Gorenstein ring, then

  $$\mathcal{X}_A \subseteq \{\mathfrak{m}\} \qquad \text{(GTT, 2015)}$$

  If $A$ is a 2-almost Gorenstein ring with minimal multiplicity, then

  $$\{\mathfrak{m}\} \subseteq \mathcal{X}_A \subseteq \{\mathfrak{m}, \mathfrak{c}\} \qquad \text{(Goto-Isobe-T, 2020)}$$

We expect that there is a strong connection between

the behavior of Ulrich ideals and the structure of base rings.

## Problem 1.1

**Determine all the Ulrich ideals in a given CM local ring.**

## Question 1.5

**How many two-generated Ulrich ideals are contained in a given numerical semigroup ring?**

- $0 < a_1, a_2, \dots, a_\ell \in \mathbb{Z}$ s.t. $\gcd(a_1, a_2, \dots, a_\ell) = 1$
- $H = \langle a_1, a_2, \dots, a_\ell \rangle = \left\{ \sum_{i=1}^{\ell} c_i a_i \;\middle|\; 0 \le c_i \in \mathbb{Z} \text{ for all } 1 \le i \le \ell \right\}$
- $A = k[[H]] = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]] \subseteq V = k[[t]] = \overline{A}$, where $k$ is a field
- $\mathrm{c}(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for all } m \in \mathbb{Z} \text{ s.t. } m \ge n\}$

Note that $t^{\mathrm{c}(H)} V \subseteq A$.

# 2. Method of computation

## Previous Method

Let

- $(A, \mathfrak{m})$ be a Gorenstein local ring with $\dim A = 1$,
- $\mathcal{X}_A$ be the set of Ulrich ideals in $A$,
- $\mathcal{Y}_A$ be the set of birational module-finite extensions $B$ of $A$

  s.t. $B$ is a Gorenstein ring and $\mu_A(B) = 2$.

Then, there exists a bijective correspondence

$$\mathcal{X}_A \to \mathcal{Y}_A, \quad I \mapsto A^I$$

where

$$A^I = \bigcup_{n \geq 0} [I^n : I^n] = I : I.$$

Let

- $V = k[[t]]$ be the formal power series ring over a field $k$
- $A$ be a $k$-subalgebra of $V$.

We say that

$$A \text{ is a } \textit{core of } V \quad \overset{def}{\Longleftrightarrow} \quad t^c V \subseteq A \text{ for some } c \gg 0.$$

## Example 2.1

- $k[[H]]$ is a core of $V$,
- $A = k[t^2 + t^3] + t^4 V$ is core, but $A \neq k[[H]]$ for any numerical semigroup $H$.

Let $A$ be a core of $V$ and suppose $t^c V \subseteq A$ with $c \gg 0$. Then

$$k[[t^c, t^{c+1}, \dots, t^{2c-1}]] \subseteq A \subseteq V$$

so that $V$ is a birational module-finite extension of $A$.

Hence, for every core $A$ of $V$,

- $V = \overline{A}$
- $A$ is a CM complete local domain with $\dim A = 1$
- $V/\mathfrak{n} \cong A/\mathfrak{m}$

where $\mathfrak{m}$ (resp. $\mathfrak{n} = tV$) stands for the maximal ideal of $A$ (resp. $V$).

Let $\mathrm{o}(*)$ denote the $\mathfrak{n}$-adic valuation of $V$, and set

$$H = v(A) = \{\mathrm{o}(f) \mid 0 \neq f \in A\}.$$

Note that

$H = v(A)$ is symmetric $\iff$ $A$ is Gorenstein (Kunz, 1970)

Let $I$ be an Ulrich ideal of $A$ with $\mu_A(I) = 2$. Choose $f, g \in I$ s.t. $I = (f, g)$ and $I^2 = fI$. Then

$$A^I = I : I = \frac{I}{f} = A + A \cdot \frac{g}{f}$$

is a core of $V$.

## Theorem 2.2

*Let $I$ be an Ulrich ideal in $A$ with $\mu_A(I) = 2$. Then one can choose $f, g \in I$ satisfying the following conditions, where $a = \mathrm{o}(f)$, $b = \mathrm{o}(g)$, and $c = \mathrm{c}(H)$.*

(1) $I = (f, g)$ *and* $I^2 = fI$.

(2) $a, b \in H$ *and* $0 < a < b < a + c$.

(3) $b - a \notin H$, $2b - a \in H$, *and* $a = 2 \cdot \ell_A(A/I)$.

(4) *If* $a \geq c$, *then* $\mathrm{e}(A) = 2$ *and* $I = A : V$.

## Example 2.3

Let $A = k[[t^2, t^{2\ell+1}]]$ $(\ell \geq 1)$ . Then

$$\mathcal{X}_A = \{(t^{2q}, t^{2\ell+1}) \mid 1 \leq q \leq \ell\}.$$

# 3. Main theorem

## Theorem 3.1 (Main theorem)

*Let* $\ell \geq 7$ *be an integer such that* $\gcd(3, \ell) = 1$ *and set* $A = k[[t^3, t^\ell]]$.

(1) *Suppose that* $\ell = 3n+1$ *where* $n \geq 3$ *is odd. Let* $q = \frac{n-1}{2}$. *Then*

$$\begin{aligned}\mathcal{X}_A \ &= \left\{ \left( t^\ell + \sum_{j=1}^{q} \alpha_j t^{\ell+3j-1}, t^{\ell+3q+2} \right) \;\middle|\; \alpha_1, \alpha_2, \ldots, \alpha_q \in k \right\} \\ &\quad \bigcup \left\{ \left( t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \;\middle|\; 1 \leq i \leq q, \alpha_0, \ldots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}.\end{aligned}$$

(2) *Suppose that* $\ell = 3n+1$ *where* $n \geq 2$ *is even. Let* $q = \frac{n}{2}$. *Then*

$$\mathcal{X}_A = \left\{ \left( t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \;\middle|\; 1 \leq i \leq q, \alpha_0, \ldots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}.$$

## Theorem 3.1 (continued)

(3) Suppose that $\ell = 3n+2$ where $n \geq 1$ is odd. Let $q = \frac{n-1}{2}$. Then

$$\mathcal{X}_A = \left\{ \left( t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \;\middle|\; 1 \leq i \leq q, \alpha_0, \ldots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}.$$

(4) Suppose that $\ell = 3n+2$ where $n \geq 2$ is even. Let $q = \frac{n}{2}$. Then

$$\begin{aligned} \mathcal{X}_A &= \left\{ \left( t^{\ell} + \sum_{j=1}^{q} \alpha_j t^{\ell+3j-2}, t^{\ell+3q+1} \right) \;\middle|\; \alpha_1, \alpha_2, \ldots, \alpha_q \in k \right\} \\ &\bigcup \left\{ \left( t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \;\middle|\; 1 \leq i \leq q, \alpha_0, \ldots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}. \end{aligned}$$

Moreover, the coefficients $\alpha_i$'s in the system of generators of $I \in \mathcal{X}_A$ are uniquely determined for $I$.

We denote by $\mathcal{X}_A^g$ the set of Ulrich ideals in $A$ generated by monomials in $t$.

## Corollary 3.2

*Let* $\ell \geq 7$ *be an integer s.t.* $\gcd(3, \ell) = 1$ *and set* $A = k[[t^3, t^\ell]]$. *Then*

(1) $\mathcal{X}_A \neq \emptyset$.

(2) $\mathcal{X}_A$ *is finite* $\iff$ $k$ *is a finite field.*

(3) $\mathcal{X}_A^g = \emptyset$ $\iff$ $\ell = 3n+1$ *or* $\ell = 3n+2$ *for some even integer* $n \geq 2$

## Example 3.3

Let $A = k[[t^3, t^7]]$. Then

$$\mathcal{X}_A = \{(t^6 + \alpha t^7, t^{10}) \mid 0 \neq \alpha \in k\}.$$

Hence, $^\#\mathcal{X}_A = {}^\# k - 1$ and $A$ does not contain monomial Ulrich ideals.

# 4. More examples

## Example 4.1

We have

$$\begin{aligned}
\mathcal{X}_{k[[t^4,t^{13}]]} &= \{(t^{12}+2\beta t^{17}+\alpha t^{26}, t^{21}+\beta t^{26}) \mid \alpha,\beta\in k,\ \beta\neq 0\}\\
&\bigcup \{(t^{16}+2\beta t^{17}+\alpha_2 t^{21}+\alpha_3 t^{26}, t^{25}+\beta t^{26}) \mid \alpha_2,\alpha_3,\beta\in k,\ \beta\neq 0\}\\
&\bigcup \{(t^{4}+\alpha t^{13}, t^{26}) \mid \alpha\in k\}\\
&\bigcup \left\{(t^{8}+\alpha_1 t^{13}+\alpha_2 t^{17}, t^{26}) \mid \alpha_1,\alpha_2\in k\right\}\\
&\bigcup \{(t^{12}+\alpha_1 t^{13}+\alpha_2 t^{17}+\alpha_3 t^{21}, t^{26}) \mid \alpha_1,\alpha_2,\alpha_3\in k\}\\
&\bigcup \{(t^{16}+\alpha_1 t^{17}+\alpha_2 t^{21}+\alpha_3 t^{25}, t^{26}) \mid \alpha_1,\alpha_2,\alpha_3\in k\}\\
&\bigcup \{(t^{20}+\alpha_1 t^{21}+\alpha_2 t^{25}+\alpha_3 t^{29}, t^{26}+\beta t^{29}) \mid \alpha_1,\alpha_2,\alpha_3,\beta\in k,\ \alpha_1^3=2\beta\}\\
&\bigcup \{(t^{24}+\alpha_1 t^{25}+\alpha_2 t^{29}+\alpha_3 t^{33}, t^{26}+\beta_1 t^{29}+\beta_2 t^{33}) \mid \alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2\in k,\\
&\qquad \alpha_1=0 \text{ if } \operatorname{ch} k=2;\ \alpha_1=\alpha_2=\beta_1=\beta_2=0 \text{ if } \operatorname{ch} k\neq 2\}.
\end{aligned}$$

For each $I\in\mathcal{X}_{k[[t^4,t^{13}]]}$, the elements of $k$ which appear in the listed expression are uniquely determined by $I$.

# 5. three-generated numerical semigroup rings

- $0 < a, b, c \in \mathbb{Z}$ s.t. $\gcd(a, b, c) = 1$ and set $H = \langle a, b, c \rangle$
- $A = k[[H]] = k[[t^a, t^b, t^c]] \subseteq V = k[[t]]$
- $\mathfrak{m} = (t^a, t^b, t^c)$

For a finitely generated $A$-module $M$, let

$$P_M^A(t) = \sum_{n=0}^{\infty} \beta_n^A(M) t^n \in \mathbb{Z}[[t]]$$

where $\beta_n^A(M)$ denotes the $n$-th Betti number of $M$.

### Theorem 5.1

*Suppose that $A = k[[H]]$ is not a Gorenstein ring. Then*

$$\beta_n^A(A/\mathfrak{m}) = \begin{cases} 1 & (n = 0) \\ 3 \cdot 2^{n-1} & (n > 0) \end{cases} \quad \textit{and} \quad P_{A/\mathfrak{m}}^A(t) = \frac{1+t}{1-2t}.$$

### Corollary 5.2 (cf. Gasharov-Peeva-Welker, 2000)

*Every three-generated non-Gorenstein numerical semigroup ring is Golod.*

Note that

- every Golod local ring which is not a hypersurface must be $G$-regular. (Avramov-Martsinkovsky, 2002)

### Corollary 5.3

*Every three-generated non-Gorenstein numerical semigroup ring contains no Ulrich ideals generated by two elements.*

**Thank you for your attention.**